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Analytical positon, negaton and complexiton solutions for the coupled modified KdV system

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Abstract
New analytical positon, negaton and complexiton solutions for the coupled modified KdV system are constructed by means of the Darboux transformation with a constant seed solution and given out both analytically and graphically.

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1. Introduction

As is well known, a new classification of exact and explicit solutions of integrable models was proposed recently according to the property of associated spectral parameters [1]. Negatons, related to the negative spectral parameter, are usually expressed by hyperbolic functions and positons are expressed by means of trigonometric functions related to the positive spectral parameters. For various important integrable systems such as the Korteweg–de Vries (KdV) equation, there is no non-singular positon. But negatons can be both singular and nonsingular. Especially, the non-singular negatons are just the usual solitons which have been studied extensively. The so-called complexiton, which is related to the complex spectral parameters, is expressed by the combinations of trigonometric functions and hyperbolic functions. The known complexiton solutions for (1 + 1)-dimensional integrable functions are singular, but some types of analytical complexitons in (2 + 1)- and (3 + 1)-dimensions can easily be obtained because of the existence of arbitrary functions in the expression of exact solutions [2–4].

For other integrable systems, complexiton solutions have been presented by means of different approaches. For example, using the Casoratian technique, the authors of [5] constructed the complexiton solutions to the Toda lattice equation through the Casoratian formulation. The new rational solutions, solitons, positons, negatons and complexiton solutions for the KdV equation are given by the Wronskian formula with the help of its bilinear form [6]. Professor Ma provided the complexiton solutions of the KdV equation and

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the Toda lattice equation through the Wronskian and Casoratian techniques [7]. The authors of [8] presented the positons, negatons and complexitons and their interaction solutions for the Boussinesq equation through its Wronskian determinant.

On the other hand, the coupled integrable systems, which have attracted more and more attention from the mathematicians and physicists, have also been studied extensively since the first coupled KdV system was put forward by Hirota and Satsuma [9]. Since then, many other coupled KdV systems are constructed such as Ito's system [10], the Nutku–Oğuz model [11], and so on.

Recently, new positon, negaton and complexiton solutions for the coupled KdV system and the Hirota–Satsuma coupled KdV system are presented by means of the Darboux transformation [12, 13]. These new positon, negaton and complexiton solutions are analytical or singular for different integrable systems with constant seed solution. For other coupled integrable systems, a quite general type of the coupled variable coefficient modified KdV system is derived from a two-layer fluid model by the reductive perturbation method [14]. The general coupled variable coefficient modified KdV system is written as

$$\begin{aligned} A_{1T} + r_1 A_{1XXX} &+ \left(r_2 A_1^2 + r_3 A_1 A_2 + r_4 A_1 + r_5 A_2^2 + r_6 A_2 + r_7\right) A_{1X} \\ &+ \left(r_8 A_1^2 + r_9 A_1 A_2 + r_{10} A_1 + r_{11} A_2 + r_{12}\right) A_{2X} + r_{13} A_1 = 0, \end{aligned} \tag{1}$$

$$\begin{aligned} A_{2T} + e_1 A_{2XXX} &+ \left(e_2 A_2^2 + e_3 A_1 A_2 + e_4 A_2 + e_5 A_1^2 + e_6 A_1 + e_7\right) A_{2X} \\ &+ \left(e_8 A_2^2 + e_9 A_1 A_2 + e_{10} A_2 + e_{11} A_1 + e_{12}\right) A_{1X} + e_{13} A_2 = 0. \end{aligned} \tag{2}$$

Here A_1, A_2 are two functions of X, T with the coefficients r_i and e_i $(i = 1, 2, 3, \ldots, 13)$ being arbitrary functions of T. Selecting the arbitrary functions r_i and e_i properly, one can obtain lots of the coupled modified KdV systems. For example, a special kind of the coupled modified KdV system has been presented in [15] and the multisoliton solutions have been constructed by Pfaffians.

In this paper, we do not discuss the general coupled modified KdV system but a special case of (1) and (2) for simplicity. In other words, the usual integrable modified KdV equation in $(1 + 1)$ dimensions reads

$$U_t + 6U^2 U_x + U_{xxx} = 0. \tag{3}$$

Substituting $U = u + Iv$ into (3) and separating the real and imaginary parts of (3) where $I^2 = -1$, we obtain a special type of the coupled modified KdV system

$$u_t - 12uvv_x + 6u^2 u_x - 6v^2 u_x + u_{xxx} = 0, \tag{4}$$

$$v_t + 12uvu_x + 6u^2 v_x - 6v^2 v_x + v_{xxx} = 0. \tag{5}$$

It is easy to check that (4) and (5) are just a special case of (1) and (2). For another type of the coupled modified KdV system, which can be obtained by changing U into IU, the authors of [16] constructed the complexiton solutions by Hirota's bilinear method and discussed their evolution structures in detail.

In the remaining part of the paper, we try to construct the new analytical positon, negaton and complexiton solutions for (4) and (5) by means of the Darboux transformation, which is one of the most effective tools to construct the exact and explicit solutions of the nonlinear evolution equations. In section 2, the Darboux transformation for the coupled modified KdV system is constructed from those of the usual modified KdV equation. In section 3, analytical positon, negaton and complexiton solutions for the coupled modified KdV system are given out directly from the Darboux transformation with constant seed solution. Section 4 presented the summary and discussions.

2. Darboux transformation for the coupled modified KdV system

In this section, we apply the Darboux transformation to the coupled modified KdV system (4) and (5) in order to construct its exact solutions. Because the coupled modified KdV system is obtained by imposing the complex variables on the usual modified KdV equation (3), we can construct the Darboux transformation for the coupled modified KdV system (4) and (5) from those of the modified KdV equation easily.

It is known that the Lax pair for the modified KdV equation (3) reads

$$\Phi_{1x} = \lambda\Phi_1 + U\Phi_2, \tag{6}$$

$$\Phi_{2x} = -\lambda\Phi_2 - U\Phi_1 \tag{7}$$

and

$$\Phi_{1t} = -(4\lambda^3 + 2\lambda U^2)\Phi_1 - (4\lambda^2 U + 2\lambda U_x + 2U^3 + U_{xx})\Phi_2, \tag{8}$$

$$\Phi_{2t} = (4\lambda^2 U - 2\lambda U_x + 2U^3 + U_{xx})\Phi_1 + (4\lambda^3 + 2\lambda U^2)\Phi_2. \tag{9}$$

If two functions $\{f_1, f_2\}$ satisfy (6)–(9) when $\lambda = \lambda_0$, then the new potential function

$$\widetilde{U} = -U - \frac{4\lambda_0 f_1 f_2}{f_1^2 + f_2^2} \tag{10}$$

also satisfies the modified KdV equation (3) and corresponding wavefunctions transformation becomes

$$\widetilde{\Phi_1} = \left(\lambda - \lambda_0 \frac{f_1^2 - f_2^2}{f_1^2 + f_2^2}\right)\Phi_1 - \frac{2\lambda_0 f_1 f_2}{f_1^2 + f_2^2}\Phi_2, \tag{11}$$

$$\widetilde{\Phi_2} = \frac{2\lambda_0 f_1 f_2}{f_1^2 + f_2^2}\Phi_1 - \left(\lambda + \lambda_0 \frac{f_1^2 - f_2^2}{f_1^2 + f_2^2}\right)\Phi_2. \tag{12}$$

Then transformation (10)–(12) is just the Darboux transformation for the modified KdV equation (3).

Substituting $\Phi_1 = \phi_1 + I\phi_2$, $\Phi_2 = \varphi_1 + I\varphi_2$ into (6)–(9) and noting $U = u + Iv$, the Lax pair for (4) and (5) is as follows:

$$\phi_{1x} = \lambda\phi_1 + u\varphi_1 - v\varphi_2, \tag{13}$$

$$\phi_{2x} = \lambda\phi_2 + u\varphi_2 + v\varphi_1, \tag{14}$$

$$\varphi_{1x} = -\lambda\varphi_1 - u\phi_1 + v\phi_2, \tag{15}$$

$$\varphi_{2x} = -\lambda\varphi_2 - u\phi_2 - v\phi_1, \tag{16}$$

$$\begin{aligned}\phi_{1t} = {} & (2\lambda v^2 - 4\lambda^3 - 2\lambda u^2)\phi_1 + 4\lambda uv\phi_2 + (6uv^2 - 2u^3 - 4\lambda^2 u - 2\lambda u_x - u_{xx})\varphi_1 \\ & + (2\lambda v_x + v_{xx} + 4\lambda^2 v + 6u^2 v - 2v^3)\varphi_2,\end{aligned} \tag{17}$$

$$\begin{aligned}\phi_{2t} = {} & -4\lambda uv\phi_1 + (2\lambda v^2 - 4\lambda^3 - 2\lambda u^2)\phi_2 + (2v^3 - 4\lambda^2 v - v_{xx} - 6u^2 v - 2\lambda v_x)\varphi_1 \\ & + (6uv^2 - 2u^3 - u_{xx} - 4\lambda^2 u - 2\lambda u_x)\varphi_2,\end{aligned} \tag{18}$$

$$\begin{aligned}\varphi_{1t} = {} & (4\lambda^3 - 2\lambda v^2 + 2\lambda u^2)\varphi_1 - 4\lambda uv\varphi_2 + (4\lambda^2 u + u_{xx} + 2u^3 - 2\lambda u_x - 6uv^2)\phi_1 \\ & + (2v^3 - 6u^2 v + 2\lambda v_x - 4\lambda^2 v - v_{xx})\phi_2,\end{aligned} \tag{19}$$

$$\varphi_{2t} = 4\lambda uv\varphi_1 + (4\lambda^3 - 2\lambda v^2 + 2\lambda u^2)\varphi_2 + (4\lambda^2 v + v_{xx} - 2\lambda v_x + 6u^2 v - 2v^3)\phi_1 + (4\lambda^2 u + u_{xx} + 2u^3 - 2\lambda u_x - 6uv^2)\phi_2. \tag{20}$$

Then it is easy to prove that the compatibility conditions of (13)–(20) are just the coupled modified KdV system (4) and (5). Though the Lax pair for the coupled modified KdV system seems very complicated, we can still construct its Darboux transformation from those of the usual modified KdV equation which has been expressed by (10)–(12).

With the help of the Darboux transformation for the modified KdV equation (10)–(12), we can obtain the Darboux transformation for the coupled modified KdV system (4) and (5) as follows:

$$\widetilde{u} = -u - \frac{4\lambda_0 AC}{F}, \tag{21}$$

$$\widetilde{v} = -v - \frac{4\lambda_0 BD}{F}, \tag{22}$$

$$\begin{pmatrix} \widetilde{\phi_1} \\ \widetilde{\phi_2} \\ \widetilde{\varphi_1} \\ \widetilde{\varphi_2} \end{pmatrix} = \frac{1}{F} \begin{pmatrix} \lambda - \lambda_0 CD & 4\lambda_0 AB & -2\lambda_0 AC & -2\lambda_0 BD \\ -4\lambda_0 AB & \lambda - \lambda_0 CD & 2\lambda_0 BD & -2\lambda_0 AC \\ -4\lambda_0 AB & \lambda - \lambda_0 CD & 2\lambda_0 BD & -2\lambda_0 AC \\ -2\lambda_0 BD & 2\lambda_0 AC & -4\lambda_0 AB & -\lambda - \lambda_0 CD \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{23}$$

where

$$A \equiv A(f_{11}, f_{12}, f_{21}, f_{22}) = f_{11}f_{21} + f_{12}f_{22},$$
$$B \equiv B(f_{11}, f_{12}, f_{21}, f_{22}) = f_{12}f_{21} - f_{11}f_{22},$$
$$C \equiv C(f_{11}, f_{12}, f_{21}, f_{22}) = f_{11}^2 + f_{12}^2 + f_{21}^2 + f_{22}^2,$$
$$D \equiv D(f_{11}, f_{12}, f_{21}, f_{22}) = f_{11}^2 + f_{12}^2 - f_{21}^2 - f_{22}^2,$$
$$F \equiv F(f_{11}, f_{12}, f_{21}, f_{22}) = [(f_{12} + f_{21})^2 + (f_{11} - f_{22})^2][(f_{12} - f_{21})^2 + (f_{11} + f_{22})^2],$$

and $\{f_{11}, f_{12}, f_{21}, f_{22}\}$ are solutions of the Lax pair (13)–(20) with $\lambda = \lambda_0, \phi_1 = f_{11}, \phi_2 = f_{12}, \varphi_1 = f_{21}, \varphi_2 = f_{22}$. Then (21)–(23) are just the Darboux transformation for the coupled modified KdV system.

3. Positon, negaton and complexiton solutions

In this section, we study the positon, negaton and complexiton solutions for the coupled modified KdV system (4) and (5) by means of its Darboux transformation (21)–(23) which have been given in section 2.

3.1. Positon solution, $\lambda > 0$

Based on the Darboux transformation in the last section, the analytical positon solutions for the coupled modified KdV system can be constructed easily. Taking the seed solution as $\{u = 0, v = 0\}$ and solving the Lax pair (13)–(20) directly, one can find that

$$f_{11} = c_1 e^{\lambda x - 4\lambda^3 t}, \qquad f_{12} = c_2 e^{\lambda x - 4\lambda^3 t}, \qquad f_{21} = c_3 e^{4\lambda^3 t - \lambda x}, \qquad f_{22} = c_4 e^{4\lambda^3 t - \lambda x}, \tag{24}$$

where c_1, c_2, c_3, c_4 are four arbitrary constants. Substituting (24) into (21) and (22) with the zero seed solution and the constant parameter selections

$$c_1 = -1, \qquad c_2 = -4, \qquad c_3 = 2, \qquad c_4 = 3, \qquad \lambda_0 = \frac{1}{2}, \tag{25}$$

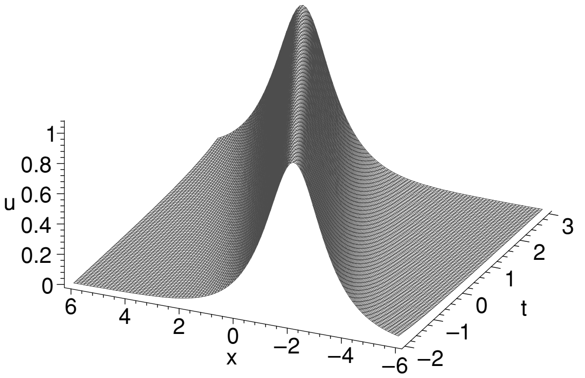

Figure 1. Structure of the positon solution for the field u expressed by (26).

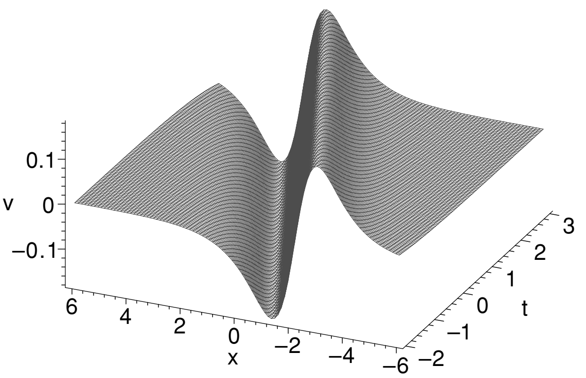

Figure 2. Positon solution for the field v given by (27).

one can obtain the analytical positon solutions for the coupled modified KdV system as follows:

$$u = \frac{476e^{x-t} + 364e^{t-x}}{289e^{2x-2t} + 169e^{2t-2x} + 342}, \tag{26}$$

$$v = \frac{-170e^{x-t} + 130e^{t-x}}{289e^{2x-2t} + 169e^{2t-2x} + 342}. \tag{27}$$

Figures 1 and 2 show the structures of the analytical positon solutions for u and v, respectively.

3.2. *Negaton solution,* $\lambda < 0$

In order to obtain the negaton solution for the coupled modified KdV system, one only needs to substitute (24) into (21) and (22) with the spectral parameter $\lambda < 0$.

Selecting the parameters as

$$c_1 = 7, \qquad c_2 = -3, \qquad c_3 = 1, \qquad c_4 = -6, \qquad \lambda_0 = -1, \tag{28}$$

one can obtain the analytical negaton solutions for the coupled modified KdV system

$$u = \frac{5800e^{8t-2x} + 3700e^{2x-8t}}{3364e^{16t-4x} + 1369e^{4x-16t} - 1792}, \tag{29}$$

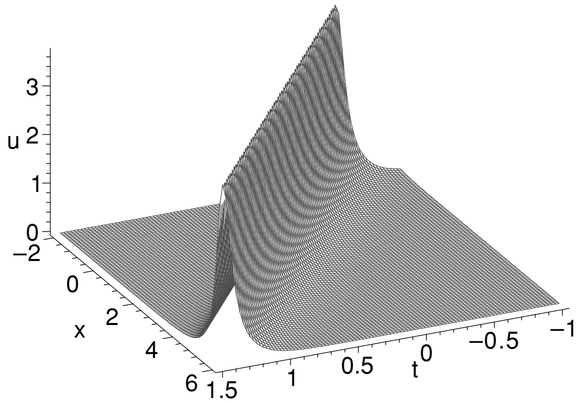

Figure 3. Analytical negaton solution for the field u given by (29).

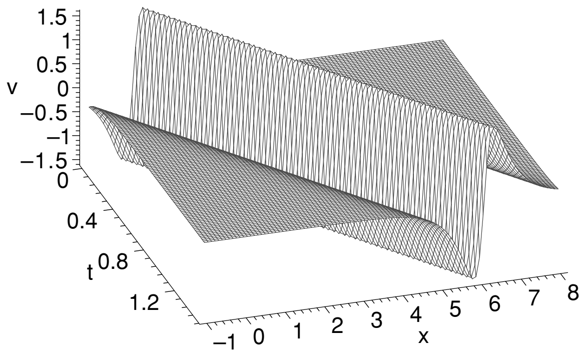

Figure 4. Analytical negaton solution for the field v given by (30).

$$v = \frac{5772\mathrm{e}^{2x-8t} - 9048\mathrm{e}^{8t-2x}}{3364\mathrm{e}^{16t-4x} + 1369\mathrm{e}^{4x-16t} - 1792}, \tag{30}$$

which are shown in figures 3 and 4.

3.3. Complexiton solutions

For many integrable systems, complexiton solutions have been found in [1, 5] and most of the known complexiton solutions have singularities, such as the KdV equation, the Boussinesq equation and the Toda lattice equation. The analytical complexiton solutions for the integrable system have been found for the sine-Gordon equation by the iteration of the Bäcklund transformation.

Fixing the spectral parameter λ as a complex number in (24), one can easily obtain the complexiton solutions for the coupled modified KdV system from the Darboux transformation. Selecting the parameters

$$c_1 = 2, \qquad c_2 = 2, \qquad c_3 = 3, \qquad c_4 = 5, \qquad \lambda_0 = I, \tag{31}$$

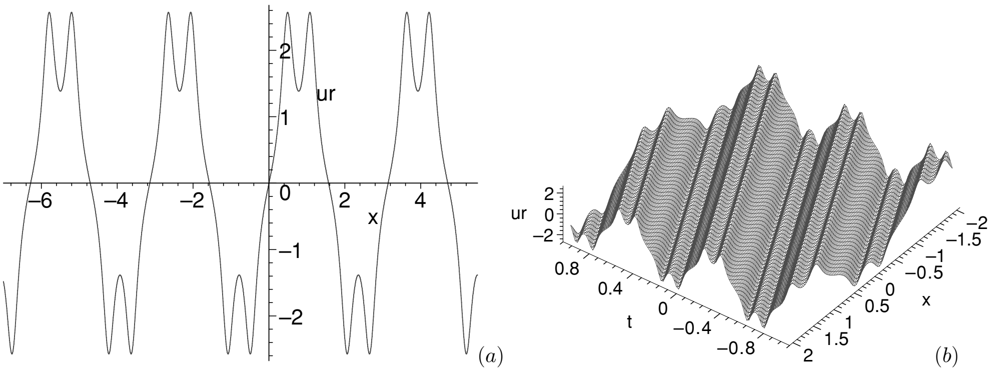

Figure 5. (*a*) Real part of the complexiton solution for the field u expressed by (32) at time $t = 0$; (*b*) the evolution plot.

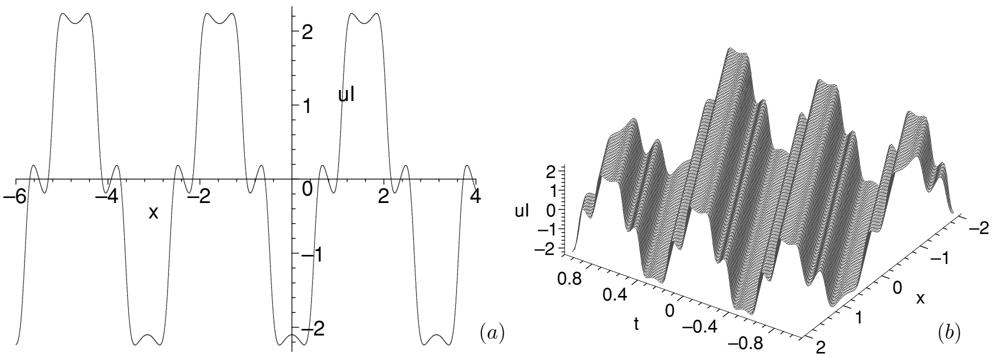

Figure 6. (*a*) Imaginary part of the complexiton solution for the field u expressed by (32) at time $t = 0$; (*b*) the evolution plot.

in (24), then from the Darboux transformation (21) and (22), the two potential fields u and v of the coupled modified KdV system become

$$u = \frac{[168 - 336\cos(4t + x)^2]I - 64\sin(8t + 2x)}{[84\sin(8t + 2x) - 336\cos(4t + x)^3\sin(4t + x)]I - \frac{505}{4}\sin(2x + 8t)^2 + 80}, \tag{32}$$

$$v = \frac{[88\cos(4t + x)^2 - 44]I + 231\cos(4t + x)\sin(4t + x)}{[84\sin(8t + 2x) - 336\cos(4t + x)^3\sin(4t + x)]I - \frac{505}{4}\sin(2x + 8t)^2 + 80}. \tag{33}$$

It seems that (32) and (33) are the complex solutions under the parameter selections (31) for the coupled modified KdV system. Separating the real and imaginary parts of the complex solution (32) and (33), we can give the detailed structures of the complexiton solutions of the coupled modified KdV system.

From figures 5 and 6, we know that the real and imaginary parts of the complexiton solutions for the potential field u are just a special nonsingular periodic wave solution. Similarly, the real and imaginary parts of the potential field v can also be regarded as a special periodic wave solution which have been shown in figures 7 and 8.

In this section, selecting the four arbitrary constants $\{c_1, c_2, c_3, c_4\}$ and the spectral parameter λ as a real or complex number in the wavefunctions (24) with zero seed solution

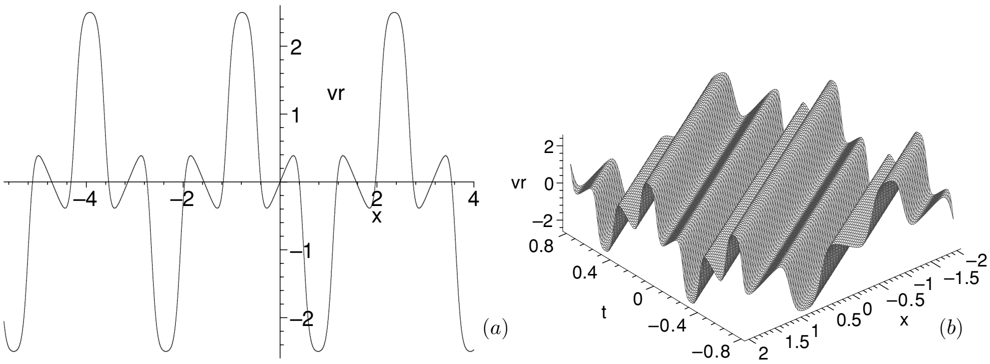

Figure 7. (*a*) Real part of the complexiton solution for the field v expressed by (33) at time $t = 0$; (*b*) the evolution plot.

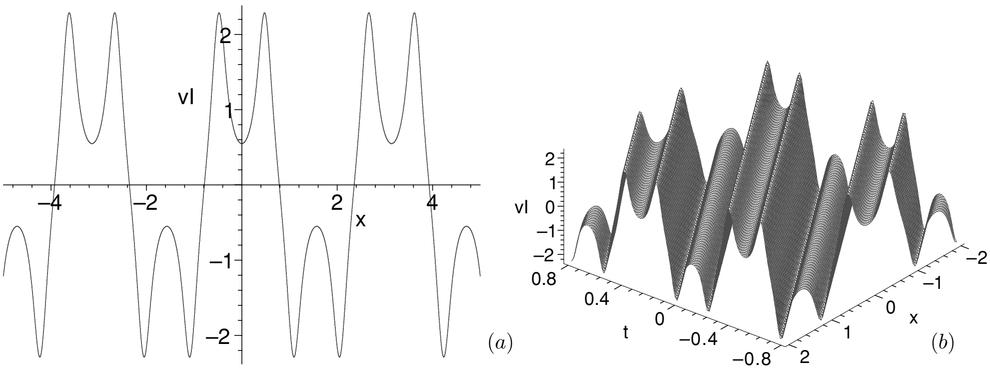

Figure 8. (*a*) Imaginary part of the complexiton solution for the field v expressed by (33) at time $t = 0$; (*b*) the evolution plot.

properly, we obtain the analytical positon, negaton and complexiton solutions for the coupled modified KdV system from its Darboux transformation given in section 2.

4. Summary and discussion

In this paper, exact solutions of a coupled modified KdV system which is obtained as a special case of the general coupled variable coefficient modified KdV system derived from the two layer fluid model are studied in detail by means of the Darboux transformation. This coupled modified KdV system can also be read off from the well-known modified KdV equation by assuming its field complex. The Lax pair and the Darboux transformation for the coupled modified KdV system can be obtained easily thanks to those of the modified KdV equation.

It is known that the Darboux transformation for the modified KdV equation can be iterated step by step by introducing the called Darboux matrix, but we do not give the compact expression of the general N-step Darboux transformation for the coupled modified KdV system because the first step Darboux transformation has such a complicated expression. How to write the N-step Darboux transformation for the coupled modified KdV system in a compact expression such as Wronskian or Grammian determinant will be studied further in a separate work.

Starting from the trivial zero seed solution, the analytical positon, negaton and complexiton solutions for the coupled modified KdV system are constructed from the Darboux transformation directly which are related to the real or complex spectral parameters. The positon and negaton solutions of the coupled modified KdV system can be seen as soliton-like solutions and the real or imaginary parts of the complexiton solutions are just special cases of the periodic solutions of the integrable system. It is important to point out that the positon, negaton and complexiton solutions obtained in this paper are nonsingular which are not like those of the single modified KdV equation. And if the seed solution of the Darboux transformation is not zero, we can still construct many more types of the analytical positon, negaton and complexiton solutions for the coupled modified KdV system. These new positon, negaton and complexiton solutions from the Darboux transformation may be a little different with the exact solutions in real applications such as in fluid mechanics and aerobiologist because of omitting many boundary conditions.

For the usual modified KdV equation, many more types of the exact solutions, such as the rational solution, periodic solution and breather solution, have been studied extensively by many authors under different approaches such as the Bäcklund transformation and the Hirota bilinear form. The similar solution for the coupled modified KdV system can be obtained easily in similar ways, but the interaction behavior of these solutions must be quite different. How to construct the Bäcklund transformation for the coupled modified KdV system in order to give out the corresponding positon, negaton and complexiton solutions is also an interesting and a significant topic. The more about this coupled model and the applications of the analytical positon, negaton and complexiton solutions proposed in the paper are worthy of studying further.

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References

[1] Ma W X 2002 *Phys. Lett.* A **301** 35
[2] Lou S Y, Hu H C and Tang X Y 2005 *Phys. Rev.* E **71** 036604
[3] Tang X Y, Lou S Y and Zhang Y 2002 *Phys. Rev.* E **66** 046601
[4] Hu H C and Lou S Y 2005 *Phys. Lett.* A **341** 422
[5] Ma W X and Maruno K 2004 *Physica* A **343** 219
[6] Ma W X and You Y 2005 *Trans. Am. Math. Soc.* **357** 1753
[7] Ma W X 2005 *Nonlinear Anal.* **63** e2461
[8] Li C X, Ma W X, Liu X J and Zeng Y B 2007 *Inverse Problems* **23** 279
[9] Hirota R and Satsuma J 1981 *Phys. Lett.* A **85** 407
[10] Ito M 1982 *Phys. Lett.* A **91** 335
[11] Nutku Y and Oğuz O 1990 *Nuovo Cimento Soc. Ital. Fis.* B **105** 1381
[12] Hu H C, Tong B and Lou S Y 2006 *Phys. Lett.* A **351** 403
[13] Hu H C and Liu Y 2008 *Phys. Lett.* A **372** 5795
[14] Gao Y and Tang X Y 2007 *Commun. Theor. Phys.* **48** 961
[15] Iwao M and Hirota R 1997 *J. Phys. Soc. Japan* **66** 577
[16] Yang J R and Mao J J 2008 *Chin. Phys. Lett.* **25** 1527